NBSIR 73-243

On the Compression of a Cylinder in Contact with a Plane Surface

B. Nelson Norden

Institute for Basic Standards
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Final Report

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NOMENCLATURE

P_o	maximum pressure at center of contact zone
a	major axis of ellipse of contact
b	minor axis of ellipse of contact (also half-width of contact in the cylinder-plane case)
R_1, R_1'	principal radii of curvature of body 1
R_2, R_2'	principal radii of curvature of body 2
Z_1	distance from a point on body 1 to the undeformed condition point of body 2
Z_2	same terms as above for body 2 to body 1
W_1	deformation of a point on body 1
W_2	deformation of a point on body 2
δ	total deformation of bodies 1 and 2
ω	angle between planes $x_1 z$ and $x_2 z$
λ_i	parameter equal to $\frac{1 - \nu i^2}{\pi E_i}$ where νi is Poisson's ratio for body i and Ei is Young's modulus for body i
Φ	potential function at any point on the surface
e	eccentricity of the ellipse of contact $e^2 = 1 - \frac{b^2}{a^2}$
P	total load applied to produce deformation
K	complete elliptic integral of the first kind
E	complete elliptic integral of the second kind
σ	stress at some point on surface of body
h	square of the complementary modulus or $(1 - e^2)$
V	potential function used by Lundberg
M	mutual approach of remote points in two plates with cylinder between the two plates

ABSTRACT

The measurement of a diameter of a cylinder has widespread application in the metrology field and industrial sector. Since the cylinder is usually placed between two flat parallel anvils, one needs to be able to apply corrections, to account for the finite measuring force used, for the most accurate determination of a diameter of the cylinder.

An extensive literature search was conducted to assemble the equations which have been developed for deformation of a cylinder to plane contact case. There are a number of formulae depending upon the assumptions made in the development. It was immediately evident that this subject has been unexplored in depth by the metrology community, and thus no coherent treatise for practical usage has been developed.

This report is an attempt to analyze the majority of these equations and to compare their results within the force range normally encountered in the metrology field. Graphs have been developed to facilitate easy computation of the maximum compressive stress encountered in the steel cylinder-steel plane contact case and the actual deformation involved.

Since the ultimate usefulness of any formula depends upon experimental verification, we have compiled results of pertinent experiments and various empirical formulae. A complete bibliography has been included for the cylinder-plane contact case for the interested reader.

INTRODUCTION

The problem of contact between elastic bodies (male, female or neuter gender) has long been of considerable interest. Assume that two elastic solids are brought into contact at a point O as in Figure 1. If collinear forces are applied so as to press the two solids together, deformation occurs, and we expect a small contact area to replace the point of the unloaded state. If we determine the size and shape of this contact area and the distribution of normal pressure, then the interval stresses and deformation can be calculated.

The mathematical theory for the general three-dimensional contact problem was first developed by Hertz in 1881. The assumptions made are:

1) the contacting surfaces are perfectly smooth so that the actual shape can be described by a second degree equation of the form

$$z = Dx^2 + Ey^2 + Fxy.$$

2) The elastic limits of the materials are not exceeded during contact. If this occurs, then permanent deformation to the materials occurs.

3) The two bodies under examination must be isotopic.

4) Only forces which act normal to the contacting surfaces are considered. This means that there is assumed to be no frictional forces at work within the contact area.

5) The other assumption is that the contacting surfaces must be small in comparison to the entire surfaces.

Based on the above assumptions and by applying potential theory, Hertz showed that:

1) the contact area is bounded by an ellipse whose semiaxes can be calculated from the geometric parameters of the contacting bodies.

2) The normal pressure distribution over this area is:

$$p_o \left[1 - (x/a)^2 - (y/b)^2\right]^{1/2}$$

where p_o = maximum pressure at center

a = major axis of ellipse of contact

b = minor axis of ellipse.

The above assumptions are valid in the field of dimensional metrology because the materials (usually possessing finely lapped surfaces), and the measuring forces normally used are sufficient for the Hertzian equations to be accurate. In the case of surfaces that are not finely lapped, the actual deformation may differ by more than 20% from those calculated from equations.

Since the subject of deformation has such widespread impact on the field of precision metrology, we have decided to publish separate reports - (1) dealing with line contact and particularly the contact of a cylinder to a plane, and; (2) which treats the general subject of contacting bodies and derives formulae for all other major cases which should be encountered in the metrology laboratory.

An exhaustive literature search was conducted to determine equations currently in use for deformation of a cylinder to flat surface. The ultimate usefulness of deformation formulae depends on their experimental verification and, while there is an enormous amount of information available for large forces, it was found that the data is scarce for forces in the range used in measurement science. One reason for this scarcity is the degree of geometric perfection required in the test apparatus and the difficulty of measuring the small deformations reliably.

Depending upon the assumptions made, there are a number of formulae in use. Various equations will be analyzed along with the assumptions inherent in their derivations. There are basically three approaches to the problem for the deformation of a cylinder with diameter D in contact with a plane over a length L and under the action of force P:

1) the approach where a solution is generated from the general three-dimensional case of curved bodies by assigning the plane surface a radius of curvature. This is the same as replacing the plane surface with a cylindrical surface with a very large radius of curvature. The area of contact is then an elongated ellipse.

2) The approach where the contact area between a cylinder and plane is assumed to be a finite rectangle of width 2b and length L where $L \gg b$.

3) The determination of compression formula by empirical means.

GENERAL DESCRIPTION OF CONTACT PROBLEM

When two homogeneous, elastic bodies are pressed together, a certain amount of deformation will occur in each body, bounded by a curve called the curve of compression. The theory was first developed by H. Hertz [1].

Figure 1 shows two general bodies in the unstressed and undeformed state with a point of contact at O. The two surfaces have a common tangent at point O. The principal radii of curvature of the surface at the point of contact is R_1 for body 1, and R_2 for body 2. R_1' and R_2' represent the other radii of curvature of each body. The radii of curvature are measured in two planes at right angles to one another. The principal radii of curvature may be positive if the center of curvature lies within the body, and negative if the center of curvature lies outside the body. Also planes $x_1 z$ and $x_2 z$ should be chosen such that

$$\left(\frac{1}{R_1} + \frac{1}{R_2}\right) > \left(\frac{1}{R_1'} + \frac{1}{R_2'}\right)$$

The angle ω is the angle between the normal sections of the two bodies which contain the principal radii of curvature R_1 and R_2.

Figure 2 shows a cross-section of the two surfaces near the point of contact O. We must limit our analysis to the case where the dimensions of the compressed area after the bodies have been pressed together are small in comparison with the radii of curvature of bodies 1 and 2. We also assume that the surface of each body near the point of contact can be approximated by a second degree equation of the form

$$Z = Dx^2 + Ey^2 + 2\ Fxy$$

where D, E and F are arbitrary constants.

If the two bodies are pressed together by applied normal forces (Figure 3), then a deformation occurs near the original point of contact along the Z-axis. Here again, we consider only forces acting parallel to the z-axis where the distance from the z-axis is small.

The displacements at a point are w_1 and w_2 where w_1 is the deformation of point P_1 of body 1 and w_2 is the deformation of point P_2 for body 2, plane C is the original plane of tangency; z_1 is the distance from P_1 to the undeformed state, and z_2 is the distance from P_2 to the undeformed state. For points inside the contact area, we have

$$(z_1 + w_1) + (z_2 + w_2) = \delta \qquad (1)$$

where δ is the total deformation which we are so diligently seeking.

The equation for surface 1 may be written as:

$$z_1 = D_1 x^2 + E_1 y^2 + 2F_1 xy$$

and for surface 2,

$$z_2 = D_2 x^2 + E_2 y^2 + 2F_2 xy.$$

Since the sum of z_1 and z_2 enter into the equation we obtain

$$z_1 + z_2 = (D_1 + D_2)x^2 + (E_1 + E_2)y^2 + 2(F_1 + F_2)xy. \qquad (2)$$

Now Hertz showed that the axis can be transformed so that $F_1 = -F_2$, and hence, the xy term vanishes. To simplify the above equation further we replace the constants $(D_1 + D_2)$ with A and $(E_1 + E_2)$ with B thus giving,

$$z_1 + z_2 = Ax^2 + By^2$$

From equation (1) we obtain:

$$Ax^2 + By^2 + w_1 + w_2 = \delta \qquad (3)$$

The constants A and B are expressible in terms of combinations of the principal curvatures of the surfaces and the angle between the planes of curvature. These combinations have been derived by Hertz and are as follows:

$$A + B = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_1'} + \frac{1}{R_2} + \frac{1}{R_2'}\right) \tag{4}$$

$$B - A = \frac{1}{2}\left[\left(\frac{1}{R_1} - \frac{1}{R_1'}\right)^2 + \left(\frac{1}{R_2} - \frac{1}{R_2'}\right)^2 + 2\left(\frac{1}{R_1} - \frac{1}{R_1'}\right)\left(\frac{1}{R_2} - \frac{1}{R_2'}\right)\cos 2\omega\right]^{1/2} \tag{5}$$

Since the points within the compressed area are in contact after the compression we have:

$$w_1 + w_2 = \delta - Ax^2 - By^2$$

and since δ is the value of w_1 and w_2 at the origin (Figure 3, $x = y = 0$), we must evaluate w_1 and w_2.

The pressure P between the bodies is the resultant of a distributed pressure (P' per unit of area), over the compressed area. From Prescott [2] the values of the deformations w_1 and w_2 under the action of normal forces are:

$$w_1 = \lambda_1 \Phi (x, y) \tag{6}$$

and,

$$w_2 = \lambda_2 \Phi (x, y) \tag{7}$$

where $$\lambda_i = \left(\frac{1 - \nu i^2}{\pi E i}\right)$$

ν_i = Poisson's ratio for the ith body

E_i = Modulus of elasticity for the ith body,

and $\Phi(x, y) = \iint_A \frac{P'}{r}\, dx'dy'$ which represents the potential at a point on the surface. Here r is the distance from some point (x, y) to another point (x', y') and P' is the surface density.

By substitution in Equation 3 we obtain,

$$(\lambda_1 + \lambda_2)\iint_A \frac{P'}{r}\, dx'\, dy' = \delta - Ax^2 - By^2 \qquad (8)$$

where the subscripts 1 and 2 represent the elastic constants for bodies 1 and 2.

One important fact should be observed from Equations 6 and 7 and this is:

$$\frac{w_1}{w_2} = \frac{\lambda_1}{\lambda_2} = \frac{\left(\frac{1 - \nu_1^2}{\pi E_1}\right)}{\left(\frac{1 - \nu_2^2}{\pi E_2}\right)} \qquad (9)$$

which means if the two bodies are made of the same material $w_1 = w_2$.

The integral equation 8 allows one to compute the contact area, the pressure distribution, and the deformation of the bodies.

The problem is now to find a distribution of pressures to satisfy equation 8. Since the formula for Φ_o is a potential function due to matter distributed over the compressed area with surface density P' we see the analogy between this problem and potential theory. Hertz saw the analogy since the integral on the left side of equation 8 is of a type commonly found in potential theory, where such integrals give the potential of a distribution of charge and the potential at a point in the interior of a uniformly charged ellipsoid is a quadratic function of the coordinates.

If an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has a uniform charge density ρ with mass $\pi\rho abc$, then the potential within the ellipse is given by Kellog [3] as

$$\Phi(x, y, z) = \pi\rho abc \int_0^\infty \left(1 - \frac{x^2}{a^2 + \Psi} - \frac{y^2}{b^2 + \Psi} - \frac{z^2}{c^2 + \Psi}\right) \frac{d\Psi}{((a^2 + \Psi)(b^2 + \Psi)(c^2 + \Psi))^{1/2}} \tag{10}$$

If we consider the case where the ellipsoid is very much flattened ($c \to 0$) then we have

$$\Phi(x, y) = \pi\rho abc \int_0^\infty \left(1 - \frac{x^2}{a^2 + \Psi} - \frac{y^2}{b^2 + \Psi}\right) \frac{d\Psi}{((a^2 + \Psi)(b^2 + \Psi)(\Psi))^{1/2}} \tag{11}$$

The potential due a mass density is

$$\sigma(x', y') = \frac{3p}{2\pi ab} \sqrt{1 - \frac{X'^2}{a^2} + \frac{y'^2}{b^2}} \tag{12}$$

distributed over the ellipse $\frac{X'^2}{a^2} + \frac{y'^2}{b^2} = 1$ in the plane $Z = 0$, where the total load P is given by,

$$P = 4/3\ \pi\rho c\ ab. \tag{13}$$

By substituting into Equation 11 we obtain,

$$\Phi(x, y) = \frac{3P}{4} \int_0^\infty \left(1 - \frac{X^2}{a^2 + \Psi} - \frac{y^2}{b^2 + \Psi}\right) \frac{d\Psi}{((a^2 + \Psi)(b^2 + \Psi)(\Psi))^{1/2}} \tag{14}$$

From Equations 8, 11 and 14 we obtain,

$$(\lambda_1 + \lambda_2) \iint_A \frac{P'}{r}\ dx'\ dy' =$$

$$(\lambda_1 + \lambda_2)\ \frac{3}{4}\ P \int_0^\infty \left(1 - \frac{X^2}{a^2 + \Psi} - \frac{y^2}{b^2 + \Psi}\right) \frac{d\Psi}{((a^2 + \Psi)(b^2 + \Psi)(\Psi))^{1/2}} \tag{15}$$

Thus we have,

$$\iint_A \frac{P'}{r}\ dx'\ dy' = \frac{3}{4}\ P \int_0^\infty \left(1 - \frac{X^2}{a^2 + \Psi} - \frac{y^2}{b^2 + \Psi}\right) \frac{d\Psi}{((a^2 + \Psi)(b^2 + \Psi)(\Psi))^{1/2}} \tag{16}$$

We now substitute into Equation 8 to obtain,

$$(\lambda_1 + \lambda_2)\ \frac{3}{4}\ P \int_o^\infty \left(1 - \frac{x^2}{a^2 + \Psi} - \frac{y^2}{b^2 + \Psi}\right) \frac{d\Psi}{((a^2 + \Psi)(b^2 + \Psi)\ (\Psi))^{1/2}} =$$

$$\delta - Ax^2 - By^2 \qquad (17)$$

Since the coefficients of 1, x^2, and y^2 must be equal in Equation 17, we have,

$$\delta = \frac{3}{4}\ P\ (\lambda_1 + \lambda_2) \int_o^\infty \frac{d\Psi}{((a^2 + \Psi)(b^2 + \Psi)(\Psi))^{1/2}} \qquad (18)$$

$$A = \frac{3}{4}\ P\ (\lambda_1 + \lambda_2) \int_o^\infty \frac{d\Psi}{(a^2 + \Psi)^{3/2}((b^2 + \Psi)(\Psi))^{1/2}} \qquad (19)$$

$$B = \frac{3}{4}\ P\ (\lambda_1 + \lambda_2) \int_o^\infty \frac{d\Psi}{(b^2 + \Psi)^{3/2}((a^2 + \Psi)(\Psi))^{1/2}} \qquad (20)$$

Equations 19 and 20 determine a and b (major and minor axis of the ellipse of contact) and equation 18 determines the total deformation δ, (or the normal approach) when a and b are known.

Since the integrals in Equations 18, 19 and 20 are somewhat cumbersome, they may be expressed in terms of complete elliptic integrals where tables are readily available. Since the eccentricity (e) of any ellipse may be expressed in terms of the major and minor axis as,

$$e^2 = 1 - \frac{b^2}{a^2} \quad \text{or} \quad e = \left(1 - \frac{b^2}{a^2}\right)^{1/2}$$

we may express Equations 18, 19 and 20 in terms of the eccentricity of the contact ellipse.

From Equation 19 we obtain,

$$A = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_o^\infty \frac{d\Psi}{(a^2 + \Psi)^{3/2} (a^2 - a^2 e^2 + \Psi)^{1/2} \Psi^{1/2}}$$

By multiplying the numerator and denominator by $(\frac{1}{a^2})^{5/2}$ and making the change of variable $a^2 \xi = \Psi$ we obtain,

$$A = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_o^\infty \frac{d\xi}{a^3 (1 + \xi)^{3/2} (1 - e^2 + \xi)^{1/2} \xi^{1/2}}$$

or

$$Aa^3 = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_o^\infty \frac{d\xi}{(1 + \xi)^{3/2} (1 - e^2 + \xi)^{1/2} \xi^{1/2}} \qquad (21)$$

From the same analysis we obtain for Equations 18 and 20,

$$Ba^3 = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_o^\infty \frac{d\xi}{(1 - e^2 + \xi)^{3/2} [\xi (1 + \xi)]^{1/2}} \qquad (22)$$

and

$$\delta a = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_o^\infty \frac{d\xi}{[\xi (1 + \xi)(1 - e^2 + \xi)]^{1/2}} \qquad (23)$$

By making the substitution $\xi = \cot^2 \theta$ [3], and $d\xi = -2 \cot \theta \csc^2 \theta \, d\theta$ where $\theta : \frac{\pi}{2}$ to 0 and ξ :0 to ∞ we obtain,

$$Aa^3 = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^o \frac{-2 \cot \theta \csc^2 \theta \, d\theta}{(1 + \cot^2 \theta)^{3/2} (\cot^2 \theta (1 - e^2 + \cot^2 \theta)^{1/2})^{1/2}}$$

and since $(1 + \cot^2 \theta) = \csc^2 \theta$

$$= \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^o \frac{-2 \csc^2 \theta}{(\csc^2 \theta)^{3/2} (\csc^2 \theta - e^2)^{1/2}} \, d\theta$$

$$= \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^{o} \frac{-2}{(\csc\theta)(\csc^2\theta - e^2)^{1/2}} d\theta$$

$$= \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^{o} \frac{-2}{(\csc\theta)[\csc^2\theta(1 - \frac{e^2}{\csc^2\theta})]^{1/2}} d\theta$$

$$= \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^{o} \frac{-2}{\csc^2\theta(1 - \frac{e^2}{\csc^2\theta})^{1/2}} d\theta$$

and since $\sin\theta = \frac{1}{\csc\theta}$,

$$Aa^3 = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^{o} \frac{-2 \sin^2\theta}{(1 - e^2 \sin^2\theta)^{1/2}} d\theta \tag{24}$$

By the same analysis we find,

$$Ba^3 = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^{o} \frac{-2 \sin^2\theta}{(1 - e^2 \sin^2\theta)^{3/2}} d\theta \tag{25}$$

and,

$$\delta a = \frac{3}{4} P (\lambda_1 + \lambda_2) \int_{\pi/2}^{o} \frac{2}{(1 - e^2 \sin^2\theta)^{1/2}} d\theta \tag{26}$$

By rearranging the above equations we obtain,

$$Aa^3 = \frac{3}{2} P (\lambda_1 + \lambda_2) \int_{o}^{\pi/2} \frac{\sin^2\theta}{(1 - e^2 \sin^2\theta)^{1/2}} d\theta \tag{27}$$

$$Ba^3 = \frac{3}{2} P (\lambda_1 + \lambda_2) \int_{o}^{\pi/2} \frac{\sin^2\theta}{(1 - e^2 \sin^2\theta)^{3/2}} d\theta \tag{28}$$

$$\delta a = \frac{3}{2} P (\lambda_1 + \lambda_2) \int_{o}^{\pi/2} \frac{1}{(1 - e^2 \sin^2\theta)^{1/2}} d\theta \tag{29}$$

Now the Legendre forms of elliptic integrals of the first and second kinds are from Boas [4],

$$F(e, \phi) = \int_o^{\phi} \frac{d\phi}{(1 - e^2\sin^2\phi)^{1/2}} \qquad \begin{cases} 0 \le e \le 1 \\ e = \sin\theta \\ 0 \le \theta \le \pi/2 \end{cases}$$

$$E(e, \phi) = \int_o^{\phi} (1 - e^2\sin^2\phi)^{1/2}\, d\phi$$

where e is the modulus and ϕ the amplitude of the elliptic integral. $e' = (1 - e^2)^{1/2}$ and is called the complementary modulus.

The complete elliptic integrals of the first and second kinds are the values of F and E for $\phi = \pi/2$ so that,

$$K = K(e) = F(e, \pi/2) = \int_o^{\pi/2} \frac{d\theta}{(1 - e^2\sin^2\theta)}$$

$$E = E(e) = E(e, \pi/2) = \int_o^{\pi/2} (1 - e^2\sin^2\theta)^{1/2}\, d\theta$$

There are numerous ways to evaluate the above integrals. Hastings [5], has polynomial approximations accurate to 2 parts in 10^8 which are of the form,

$$K(e) = [a_o + a_1m_1 + \dots + a_4m_1^4] +$$

$$[b_o + b_1m_1 + \dots + b_4m_1^4]\ \ln(1/m_1)$$

a_o = 1.38629 436112	b_o = .5
a_1 = .09666 344259	b_1 = .12498 593597
a_2 = .03590 092383	b_2 = .06880 248576
a_3 = .03742 563713	b_3 = .03328 355346
a_4 = .01451 196212	b_4 = .00441 787012

where $e = \sin^2 \alpha$ and $m_1 = \cos^2 \alpha$

and the approximation for the elliptic integral of the second kind is given by,

$$E(e) = [1 + a_1 m_1 + \ldots a_4 m_1^4] + [b_1 m_1 + \ldots b_4 m_1^4] \ln(\frac{1}{m_1})$$

a_1 = .44325 141463	b_1 = .24998 368310
a_2 = .06260 601220	b_2 = .09200 180037
a_3 = .04757 383546	b_3 = .04069 697526
a_4 = .01736 506451	b_4 = .00526 449639

Now since the complete elliptic integral of the first kind is,

$$K = \int_0^{\pi/2} (1 - e^2 \sin^2 \theta)^{-1/2} \, d\theta$$

we can obtain,

$$\frac{dK}{de} = e \int_0^{\pi/2} \frac{\sin^2 \theta}{(1 - e^2 \sin^2 \theta)^{3/2}} \, d\theta \tag{30}$$

and in a similar manner for the elliptic integral of the second kind we obtain,

$$\frac{dE}{de} = -e \int_0^{\pi/2} \frac{\sin^2 \theta}{(1 - e^2 \sin^2 \theta)^{1/2}} \, d\theta \tag{31}$$

and we obtain from Equations 27, 28 and 29,

$$Aa^3 = \frac{3}{2} P (\lambda_1 + \lambda_2) (\frac{dE}{de}) (-\frac{1}{e}) \tag{32}$$

$$Ba^3 = \frac{3}{2} P (\lambda_1 + \lambda_2) (\frac{dK}{de}) (\frac{1}{e}) \tag{33}$$

$$\delta a = \frac{3}{2} P (\lambda_1 + \lambda_2) (K) \tag{34}$$

We may rewrite Equation 32 as,

$$Aa^3 = \frac{3}{2} P (\lambda_1 + \lambda_2)\int_0^{\pi/2} \frac{1}{e^2} \left(\frac{1}{(1 - e^2\sin^2\theta)^{1/2}} - \right.$$

$$\left. \frac{1}{(1 - e^2\sin^2\theta)^{1/2}} + \frac{e^2\sin^2\theta}{(1 - e^2\sin^2\theta)^{1/2}}\right) d\theta$$

$$= \frac{3}{2} P (\lambda_1 + \lambda_2) (\frac{1}{e^2})\int_0^{\pi/2} \left[\frac{-(1 - e^2\sin^2\theta)}{(1 - e^2\sin^2\theta)^{1/2}} + \frac{1}{(1 - e^2\sin^2\theta)^{1/2}}\right] d\theta$$

$$= \frac{3}{2} P (\lambda_1 + \lambda_2) (\frac{1}{e^2})\int_0^{\pi/2} \left[\frac{1}{(1 - e^2\sin^2\theta)^{1/2}} - (1 - e^2\sin^2\theta)^{1/2}\right] d\theta$$

$$Aa^3 = \frac{3}{2} P (\lambda_1 + \lambda_2) (\frac{1}{e^2}) [K - E] \tag{35}$$

Equating equations 32 and 35 we obtain,

$$\frac{dE}{de} = \frac{1}{e} (E - K) \tag{36}$$

From equation 30

$$\frac{dK}{de} = \int_0^{\pi/2} \frac{e \sin^2\theta}{(1 - e^2\sin^2\theta)^{3/2}} d\theta$$

$$= \int_0^{\pi/2} \frac{(1 - e^2\sin^2\theta)^{1/2} (e^2\sin^2\theta)}{e(1 - e^2\sin^2\theta)^{1/2} (1 - e^2\sin^2\theta)^{3/2}} d\theta$$

$$= \int_0^{\pi/2} \frac{(1 - e^2\sin^2\theta)^{1/2} [1 - (1 - e^2\sin^2\theta)]}{e(1 - e^2\sin^2\theta)^{1/2} (1 - e^2\sin^2\theta)^{3/2}} d\theta$$

$$= \int_0^{\pi/2} \left[\frac{(1 - e^2\sin^2\theta)^{1/2}}{e(1 - e^2\sin^2\theta)^{1/2} (1 - e^2\sin^2\theta)^{3/2}} - \frac{(1 - e^2\sin^2\theta)^{1/2} (1 - e^2\sin^2\theta)}{e(1 - e^2\sin^2\theta)^{1/2} (1 - e^2\sin^2)^{3/2}}\right] d\theta$$

$$= \int_0^{\pi/2} \left[- \frac{1}{e(1 - e^2 \sin^2\theta)^{1/2}} + \frac{1}{e(1 - e^2 \sin^2\theta)^{3/2}} \right] d\theta$$

$$\frac{dK}{de} = -\frac{1}{e} K + \frac{1}{e} \int_0^{\pi/2} \frac{d\theta}{(1 - e^2 \sin^2\theta)^{3/2}}$$

The integral above may be reduced to the form

$$\frac{1}{e} \int_0^{\pi/2} \frac{d\theta}{(1 - e^2 \sin^2\theta)^{3/2}} = \frac{E}{e(1 - e^2)}$$

thus,

$$\frac{dK}{de} = -\frac{1}{e} K + \frac{E}{e(1 - e^2)} = \frac{1}{e(1 - e^2)} \left[E - (1 - e^2) K \right] \tag{37}$$

By dividing Equation 32 by 33 we obtain,

$$\frac{A}{B} = \frac{\frac{-dE}{de}}{\frac{dK}{de}} \tag{38}$$

and substituting the values in Equations 36 and 37 into Equation 38, we obtain,

$$\frac{A}{B} = \frac{-(1 - e^2)(E - K)}{E - (1 - e^2)K} \tag{39}$$

We now see that for any value of the eccentricity of the contact ellipse we can obtain values for $\frac{A}{B}$, K, and $-\frac{1}{e} \frac{dE}{de}$ which will allow us to compute the normal approach δ (deformation).

SPECIAL CASE OF LINE CONTACT

In the metrology field one fundamental measurement occurs frequently, i.e., the measurement of the diameter of a cylinder. Figure 4a shows a cross-section of the typical measurement between plane parallel anvils of a measuring machine. Since it is infinitely difficult to measure the object with zero force applied, Figure 4b shows the exaggerated resultant shape of the anvils and cylinder after a measuring force is applied. Since the customer usually desires to know the "unsquashed" diameter, certain corrections must be applied to account for the measurement process.

To solve this problem we shall make use of the expressions already developed to solve for the "pressure distribution" and size of the area of contact by allowing one axis of the ellipse of contact to become infinite. To determine the deformation, the contact area will be taken as being a finite rectangle with one side very much greater than the other.

The derivation will be for the case of a pair of cylinders with their axes parallel and is based on the works of Thomas and Hoersch [7], and Love [8]. The solution for a cylinder to plane contact can easily be obtained by allowing the radius of one of the cylinders to become infinite.

Line contact occurs when two cylinders rest on each other with their axes parallel (figure 5a), and when a cylinder rests on a plane. As the two cylinders are pressed together along their axes, the resulting pressure area is a narrow "rectangle" of width 2b and length L (assuming no taper in the cylinders). In other words, the area of contact is an elongated ellipse with the major axis of the ellipse equal to L and the eccentricity approaching unity.

The distribution of compressive stress along the width 2b of the surface of contact is represented by a semi-ellipse (figure 5c). The stress distribution over the ellipse of contact for the three-dimensional case will be remembered from Equation 12 as,

$$\sigma(x', y') = \frac{3P}{2\pi ab}\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}}$$

where $\sigma(x', y')$ represents the stress acting at any point (x', y').

Now the integrated pressure over the surface of a finite rectangle across the minor axis of the ellipse in the plane $x = 0$ (figure 5c), is

$$\overline{P} = \int_{-b}^{+b} \sigma(0, y')\, dy'$$

$$= \frac{3P}{2\pi ab}\int_{-b}^{+b}\left(1 - \frac{y'^2}{b^2}\right)^{1/2} dy'$$

$$\overline{P} = \frac{3P}{2\pi ab}\left(\frac{\pi b}{2}\right) = \frac{3P}{4a} \qquad (40)$$

Now as $a \to \infty$, let $P \to \infty$ in a manner so that P/a remains equal to a finite constant. Then the value of $\overline{P}$ is the force per unit length of the contact area. For $a = \infty$, the compressive stress at any point y is given by,

$$\sigma(y) = \frac{3P}{2\pi ab}\left(1 - \frac{y^2}{b^2}\right)^{1/2} \qquad (41)$$

and by substitution of Equation 40 into Equation 41 we obtain, since $\overline{P} = P/L$,

$$\sigma(y) = \frac{2P}{\pi Lb}\left(1 - \frac{y^2}{b^2}\right)^{1/2} \qquad (42)$$

We can now see that the maximum value for the stress within the area of contact will be at the center where $y = 0$ and is,

$$\sigma_{max} = \frac{2P}{\pi Lb} \qquad (43)$$

The maximum compressive stress is important in any contact problem because the surface area of contact is small and very high stresses may easily be obtained with relatively light loads. If σ_{max} exceeds the microplastic yield point of the material, then permanent deformation will occur. A relationship for hardened steel relating yield stress to surface finish is shown in Figure 6.

We now need to develop an expression for the width of contact b, and by knowing b, the stress at any point can be computed. The surface of each cylinder may be represented by the equations,

$$Z_1 = B_1 y^2$$

$$Z_2 = B_2 y^2$$

From Equation 20 and using the expression developed in Equation 40 we obtain,

$$B = (\lambda_1 + \lambda_2) \frac{P}{L} \int_o^\infty \frac{d\Psi}{(b^2 + \Psi)^{3/2}((a^2 + \Psi)\Psi)^{1/2}} \tag{44}$$

and since the one axis of the ellipse, $a \rightarrow \infty$, the expression becomes,

$$B = (\lambda_1 + \lambda_2) \frac{P}{L} \int_o^\infty \frac{d\Psi}{(b^2 + \Psi)^{3/2} \Psi^{1/2}} \tag{45}$$

By using the expression in Equation 42, Thomas and Hoersch evaluated the above equation to give,

$$B = 2(\lambda_1 + \lambda_2) P/Lb^2 \text{ or}$$

$$b^2 = \frac{2(\lambda_1 + \lambda_2)P}{LB} \tag{46}$$

where b is the half-width of contact,

P is the total force,

L is the contact length.

For a pair of cylinders with their axes parallel and radii r_1 and r_2, respectively, we have

$$Z_1 = B_1 y^2 = \frac{1}{2r_1} y^2$$

$$Z_2 = B_2 y^2 = \frac{1}{2r_2} y^2$$

Thus,

$$B = B_1 + B_2 = \frac{r_1 + r_2}{2r_1 r_2}$$

So

$$b^2 = \frac{4(\lambda_1 + \lambda_2)P(r_1 r_2)}{L(r_1 + r_2)} \tag{47}$$

For a cylinder on a plane surface,

$$Z_1 = \frac{1}{2r} y^2$$

$$Z_2 = \frac{1}{\infty} = 0$$

thus,

$$B = \frac{1}{2r}$$

$$b^2 = \frac{4r(\lambda_1 + \lambda_2)P}{L} \tag{48}$$

Now since our cylindrical surfaces are described by By^2 forms, we obtain from Equation 8,

$$(\lambda_1 + \lambda_2)\iint_A \frac{P(y')}{r} \, dx'dy' = \delta - By^2 \tag{49}$$

where $r = ((y - y')^2 + x'^2)^{1/2}$ and the integral applies to the finite rectangle of contact with one side (L) very much longer than the other (2b).

We now have,

$$\Phi(0, y) = \int_{-b}^{+b}\int_{-L/2}^{+L/2} \frac{P(y')}{((y - y')^2 + x'^2)^{1/2}} \, dx'dy'$$

$$= \int_{-b}^{+b} 2p(y') \int_{o}^{L/2} \frac{1}{((y - y')^2 + x'^2)^{1/2}} \, dx'dy'$$

$$= \int_{-b}^{+b} 2p(y') \ln \left[\frac{\frac{L}{2} + ((y - y')^2 + (\frac{L}{2})^2)^{1/2}}{|y - y'|} \right] dy'$$

If we assume $(\frac{L}{2})$ is very large in relation to $(y - y')$ then,

$$\Phi(0, y) = \int_{-b}^{+b} 2p(y') \ln \left(\frac{L}{y - y'} \right) dy' \qquad (50)$$

and

$$\Phi(0, 0) = \int_{-b}^{+b} 2p(y') \ln \left(\frac{L}{-y'} \right) dy' \qquad (51)$$

Let us pause a moment to recap what we are doing. Equation 49 gives us the relation between the total deformation δ to the potential for points within the contact area. Since we are interested in the maximum deformation which occurs, we need to evaluate the integral in Equation 49 at the center line or where $y = 0$. Thus, Equation 51 represents that maximum potential.

Continuing, we may rewrite Equation 51 as,

$$\Phi(0, 0) = 2 \ln L \int_{-b}^{+b} p(y')dy' - \int_{-b}^{+b} p(y') \ln(y'^2) \, dy' \qquad (52)$$

Since the force per unit length $\frac{P}{L} = \int_{-b}^{+b} p(y')dy'$ and from Equation 41 we may substitute into Equation 52 to obtain,

$$\Phi(0, 0) = 2 \frac{P}{L} \ln L - \frac{2P}{\pi bL} \int_{-b}^{+b} \left(1 - \frac{y'^2}{b^2}\right)^{1/2} \ln(y'^2)dy' \qquad (53)$$

To evaluate the integral in the above relationship, we make the substitution $\frac{y}{b} = \sin \theta$ and then $dy = b \cos \theta \, d\theta$ to obtain,

$$= \int_{-\pi/2}^{+\pi/2} (1 - \sin^2\theta)^{1/2} \; [\ln(b^2 \sin^2\theta)](b\cos\theta) \; d\theta$$

$$= b \int_{-\pi/2}^{+\pi/2} \cos^2\theta \; [\ln(b^2 \sin^2\theta)] \; d\theta$$

$$= b(2\ln b) \int_{-\pi/2}^{+\pi/2} \cos^2\theta \; d\theta + 2b \int_{-\pi/2}^{+\pi/2} \cos^2\theta \; \ln|\sin\theta| d\theta$$

we know that,

$$\int_{-\pi/2}^{+\pi/2} \cos^2\theta \; d\theta = \frac{+\pi}{2}$$

and the integral

$$I = \int_{-\pi/2}^{+\pi/2} \cos^2\theta \; \ln|\sin\theta| \, d\theta$$

has been evaluated by Birens de Haan [9], to give

$$I = \frac{-\pi}{4} \; (1 + \ln 4)$$

So we obtain the value for the total integral as,

$$= b(2\ln b) \frac{\pi}{2} + 2b \; (- \frac{\pi}{4} - \frac{\pi}{4} \ln 4),$$

$$= \pi b \; (\ln b - \frac{1}{2} + \frac{\ln 4}{2})$$

Thus by substituting into Equation 53 we obtain

$$\Phi(0, \; 0) = 2 \frac{P}{L} \ln L - \frac{2P}{\pi b L} \left[\pi b (\ln b - \frac{(1 + \ln 4)}{2}) \right]$$

$$= 2 \frac{P}{L} \left[\ln L - \ln b + \frac{1 + \ln 4}{2} \right]$$

$$\Phi(o, \; o) = 2 \frac{P}{L} \left[\ln L - \ln b + 1.193145 \right]$$

Since

$$\delta = (\lambda_1 + \lambda_2) \iint_A \frac{P(y')}{r} \; dx'dy' + By^2$$

and $\Phi(o, o)$ gives us the potential at the center of pressure zone we have,

$$\delta = (\lambda_1 + \lambda_2)\ 2\,\frac{P}{L}\left[\ln L - \ln b + 1.193145\right]$$

as the total deformation of a pair of cylinders with their axes parallel or a cylinder on a flat surface. The value of b (half-width of contact) is substituted into equation 54 to obtain the appropriate answer for any particular case.

Since equation 48 gives us the value of b as,

$$b = \left[\frac{4r(\lambda_1 + \lambda_2)P}{L}\right]^{1/2}$$

we have

$$\delta = (\lambda_1 + \lambda_2)\ 2\,\frac{P}{L}\left[\ln L - \frac{1}{2}\ln\left(\frac{4r(\lambda_1 + \lambda_2)P}{L}\right) + 1.193145\right]$$

$$= \frac{P}{L}(\lambda_1 + \lambda_2)\left[2\ln L - \ln\left(\frac{4r(\lambda_1 + \lambda_2)P}{L}\right) + 2.38629\right]$$

$$= \frac{P}{L}(\lambda_1 + \lambda_2)\left[2\ln L + \ln\left(\frac{L}{4r(\lambda_1 + \lambda_2)P}\right) + 2.38629\right]$$

$$\delta = \frac{P}{L}(\lambda_1 + \lambda_2)\left[\ln\left(\frac{L^3}{4(\lambda_1 + \lambda_2)Pr}\right) + 2.38629\right] \qquad (54)$$

where P = total measuring force

L = length of contact between plane and cylid

$\lambda_i = 1 - \upsilon_1^{\,2}/\pi E_1$

r = radius of cylinder

δ = total deformation of cylinder and plane.

Another form of Equation 54 can be obtained by not evaluating the term $\left(\frac{1 + \ln 4}{2}\right)$ which gives,

$$\delta = \frac{P}{L}(\lambda_1 + \lambda_2) \left[1.00 + \ln \frac{L^3}{(\lambda_1 + \lambda_2) Pr} \right] \tag{55}$$

Remember this formula is for the case of a cylinder to plane on only one side. If both sides are desired, such as the case in Figure 4, then

$$\delta_T = 2\delta$$

If the lengths of the lines of contact are not equal or if the material of either cylinder or plane are different, then the total deformation will be,

$$\delta_T = \delta_1 + \delta_2$$

A similar approach for the computation of the deformation of a cylinder to plane has been obtained from correspondence with Bob Fergusson [10]. The basis of the work is a paper by Airey [11]in which formulae are given for the solutions of elliptic integrals when the eccentricity (e) approaches unity.

From Equations 4 and 5 we have,

$$A + B = \frac{1}{D_1} + \frac{1}{D_1'} + \frac{1}{D_2} + \frac{1}{D_2'}$$

$$(B - A)^2 = \left(\frac{1}{D_1} - \frac{1}{D_1'}\right)^2 + \left(\frac{1}{D_2} + \frac{1}{D_2'}\right)^2 + \left(\frac{1}{D_1} - \frac{1}{D_1'}\right)\left(\frac{1}{D_2} - \frac{1}{D_2'}\right) \cos 2\omega$$

where D_1 and D_1' are the two principal diameters of body 1,

D_2 and D_2' are the two principal diameters of body 2.

In the case of a cylinder to plane, if body 1 is the cylinder and body 2 is the plane, then

D_1 = Diameter of cylinder

$D_1' = \infty$