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On Characterizing Measuring Machine Geometry

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ON CHARACTERIZING MEASURING MACHINE GEOMETRY

R. J. Hocken and B. R. Borchardt

ABSTRACT

We present a simple method for removing axis nonorthogonality and checking for length dependent scale errors in two-dimensional measurements. Use of this method requires that a two-dimensional master gage (ball or grid plate, for example) be measured in two positions which differ by a rotation of the plate 90° with respect to the measuring machine axes. The method is similar to that proposed by Reeve [1] but requires only linear least squares fitting on a small computer.

1. INTRODUCTION

Typically two-dimensional standards consist of a plate with either a grid of lines deposited on the plate or an array of spheres attached to the plate. The goal of a two-dimensional measurement is to obtain the array of coordinates of either the line intersections or the ball centers. This measurement is usually done on a coordinate measuring machine where either the plate, some locating device (microscope or LVDT probe), or a combination of the two, both gage and indicator, is moved. The coordinates are read from scales attached to the axes of motion.

In a perfect system this process gives the true coordinates, but in practice the motions are never truly rectilinear, the scales on the two axes are not identical, and the axes of motion are not orthogonal. The purpose of this paper is to describe a simple technique for checking for scale errors and nonorthogonality errors and removing such systematics from the measured coordinates. In this treatment it is assumed that the motions (x and y) are linear; thus straightness errors and errors due to yaw are assumed zero. [2] This measurement proceeds as follows. The plate is placed on the machine table and oriented so that its axes are aligned, as well as possible, with the machine axes. The coordinates are measured and normalized so that the specified plate origin has coordinates (0,0). The plate is then rotated 90°, either clockwise or counterclockwise, and the coordinates remeasured. (This rotation must be within about 10 sec of 90° for the algorithm to work. Ten seconds is the equivalent of 0.0005 inch in 10 inches of travel, a figure well within the capability of any good measuring machine.) Again, the results are normalized so the plate origin has coordinates (0,0). The two sets of coordinates are inputs to a linear least squares fit which estimates the nonorthogonality, the scale error, the difference between the actual rotation and 90°, and the average x and y offsets between the two sets of coordinates. From these results the nonorthogonality can be removed and the scale differences either averaged or removed, if there is some pressing reason to trust one scale over the other. (For instance, one might use a laser interferometer for one of the scales and the machine lead screw for the other.)

2. CALCULATIONS

Suppose the gage points on the plate can be specified by a set of vectors $(\underline{X}_i)_N$ which are the "true" coordinates. Then call the measured set of N vectors in the first position (aligned with the machine axes) $(\underline{X}_{1i})_N$. The first set of measured vectors are related to the true vectors by a matrix transformation, $\underline{\underline{A}}$. That is

$$\underline{X}'_{1i} = \underline{\underline{A}}\ \underline{X}_i \ ; \ i = 1, N \tag{1}$$

where $\underline{\underline{A}}$ is a matrix which describes the machine geometry. We call $\underline{\underline{A}}$ the machine metric. For a two-dimensional measuring machine, there are several possible and equally sensible choices for $\underline{\underline{A}}$. One choice is,

$$\underline{\underline{A}}' = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \tag{2}$$

which describes a machine with scales which are equal but in which the axes are nonorthogonal by an amount α. (α is in radians and is assumed not to be more than a few microradians). This is the metric chosen by Reeve [1] in his original paper on "multiple redundancy", though he does not use the same language to express his results. The machine metric in (2) is written so that the x axes of the gage and machine are aligned and the y axis of the machine is at an angle $90°-\alpha$. This choice is arbitrary. A slightly more complicated metric one might sensibly choose is:

$$\underline{\underline{A}}_x = \begin{pmatrix} 1+\gamma & -\alpha \\ 0 & 1 \end{pmatrix} \tag{3}$$

Here γ is a small error term that is included to take into account the fact that the scale for the x axis may be different than that for the y and that one trusts the y scale more. An equivalent representation, trusting the x scale, would be

$$\underline{\underline{A}}_y = \begin{pmatrix} 1 & -\alpha \\ 0 & 1-\gamma \end{pmatrix} \tag{4}$$

Either of these forms can be built into the model described. Suppose, however, one believes the scales are different, by an amount γ, but one has no idea which scale should be trusted most. In this case one should choose a matrix that has symmetry in the scale error. A reasonable choice is:

$$\underline{\underline{A}} = \begin{pmatrix} 1 + \frac{\gamma}{2} & -\alpha \\ 0 & 1 - \frac{\gamma}{2} \end{pmatrix} \tag{5}$$

It is shown in Appendix B, that all three of these forms, eq. 3, 4, and 5, yield identical relationships between the coordinates measured in positions 1 and 2, though not identical "best" values for the coordinates. The reason for this is simply that the numbers themselves cannot ever contain information about the true choice of scale since this is arbitrary and decided by law rather than nature. Thus, only the differences between scales may be ascertained and which one is to be termed "correct" is entirely the decision of the metrologist. Since the three more general forms for the machine metric, eqs. 3, 4, and 5, yield the same observational equations, we can work equally well with only one of them.

Beginning then with $\underline{\underline{A}}_x$, we have, from equation 1, the set of vectors (coordinates) measured in the first position. They are:

$$\underline{X}'_{1i} = \underline{\underline{A}}_x \, \underline{X}_i, \quad i = 1, N \tag{1}$$

The set of vectors measured in the second position is given by

$$\underline{X}'_{i2} = \underline{\underline{A}}_x \, \underline{\underline{B}} \, \underline{X}_i, \quad i = 1, N \tag{6}$$

where $\underline{\underline{B}}$ is the finite rotation matrix,

$$\underline{\underline{B}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{7}$$

where $\theta \simeq \frac{\pi}{2}$.

The order of $\underline{\underline{A}}$ and $\underline{\underline{B}}$ is important, because $\underline{\underline{A}}$ and $\underline{\underline{B}}$ do not commute (i.e. $\underline{\underline{AB}} \neq \underline{\underline{BA}}$). The logic behind (6) is straightforward. The true coordinates after rotation are:

$$\underline{X}_{iB} = \underline{\underline{B}}\, \underline{X}_i, \quad i = 1, N \tag{8}$$

and when these coordinates are measured on the machine the numbers obtained are:

$$\underline{X}'_{2i} = \underline{\underline{A}}_x\, \underline{X}_{iB} = \underline{\underline{A}}_x\, \underline{\underline{B}}\, \underline{X}_i, \quad i = 1, N \tag{9}$$

Equations (1) and (6) may be combined to yield

$$\underline{X}'_{1i} = \underline{\underline{A}}_x\, \underline{B}^{-1}\, \underline{\underline{A}}_x^{-1}\, \underline{\underline{X}}'_{2i} \qquad i = 1,N \tag{10}$$

which is the basic observational equation. Here the data, measured coordinates in the two positions, are related by an equation which involves the machine parameters, α and γ, and the rotation angle θ.

Equation (10) would be exactly true in the absence of error. In a measuring machine, there are, however, many errors and equation (10) is only true on the average. Also, because of the way we usually make measurements, there is probably some linear offset, independent of the machine metric, between the origins in positions 1 and 2. The normalization procedure commonly used, that of subtracting the readings at the reference coordinate, systematically biases all measurements with the error in that one reference point measurement. This bias can be assessed by including in equation (10) an offset vector $\underline{\varepsilon}$, which is assumed small, so that

$$\underline{X}'_{1i} = \underline{\underline{A}}_x\, \underline{\underline{B}}^{-1}\, \underline{\underline{A}}_x^{-1}\, \underline{X}'_{2i} + \underline{\varepsilon}, \quad i = 1, N \tag{11}$$

It is easy to show that since $\underline{\varepsilon}$ is infinitesimal, $\underline{\underline{A}}_x\underline{\varepsilon}=\underline{\varepsilon}$, so that its introduction at what appears to be the last minute is mathematically sound.

We now simplify equation (11) by noting that the finite rotation matrix $\underline{\underline{B}}(\theta)$, where $\theta = \pi/2 + \beta$, reduces to an "infinitesimal" type of matrix. That is

$$\underline{\underline{B}} = \begin{pmatrix} \cos\ (\frac{\pi}{2} + \beta) & \sin\ (\frac{\pi}{2} + \beta) \\ -\sin\ (\frac{\pi}{2} + \beta) & \cos\ (\frac{\pi}{2} + \beta) \end{pmatrix} \approx \begin{pmatrix} -\beta & 1 \\ -1 & -\beta \end{pmatrix} \tag{12}$$

if one neglects terms in β^2. Also, to the same order, the inverse of $\underline{\underline{B}}$ is

$$\underline{\underline{B}}^{-1} = \begin{pmatrix} -\beta & -1 \\ 1 & -\beta \end{pmatrix} \tag{13}$$

and the inverse of $\underline{\underline{A}}_x$ is

$$\underline{\underline{A}}_x^{-1} = \begin{pmatrix} 1-\gamma & \alpha \\ 0 & 1 \end{pmatrix} \tag{14}$$

With these first order approximations, the observational equations become:

$$X'_{1i} = -(\beta+\alpha)\quad X'_{2i} - (1-\gamma)\quad Y'_{2i} + \varepsilon_x \tag{15a}$$

and

$$Y'_{1i} = (1-\gamma)\quad X'_{2i} + (\alpha-\beta)\quad Y'_{2i} + \varepsilon_y \tag{15b}$$

where we have performed the matrix multiplications indicated in equation (11). (We emphasize here that equation (15) is exactly the same for any of the three choices of $\underline{\underline{A}}$, equations (3), (4), and (5), mentioned previously.)

To obtain a best value for the parameters $(\alpha,\beta,\gamma,\varepsilon_x,\varepsilon_y)$ we must choose them such that, on the average, equations (15) are satisfied. To do this, we introduce a modified form of the traditional chi-squared which we define as follows:

$$\chi^2 = \frac{1}{2N-5} \sum_{i=1,N} (X'_{1i} - X'_{1i}\ (\text{calc})\ )^2 + (Y'_{1i} - Y'_{1i}(\text{calc}))^2 \tag{16}$$

where X'_{1i} (calc) and Y'_{1i} (calc) represent the right hand sides of equations (15a) and (15b), respectively. A best value for the parameters will occur when the quantity chi-squared is a minimum, and furthermore, chi-squared at this minimum is just the rms standard deviation in the coordinates. (We assume here that the random errors in the x and y measurements are independent with mean 0 and variance σ^2.)

We obtain the equations for the minimum in chi-squared by partial differentiation of equation (16) with respect to each of the five parameters, setting these derivatives equal to zero. A resulting system of linear equations is:

$$\underline{\underline{D}}\ \underline{P} = \underline{C}, \tag{17}$$

where $\underline{\underline{D}}$ is a 5 x 5 matrix and $\underline{P}$ and $\underline{C}$ are column vectors. Let us denote the sums which form the matrix elements of $\underline{\underline{D}}$ by dropping the i subscript, the prime and the summation sign.

Then:

$$\underline{\underline{D}} = \begin{pmatrix} 0 & 2X_2Y_2 & Y_2^2+X_2^2 & -Y_2 & -X_2 \\ X_2^2 & X_2^2 & Y_2X_2 & X_2 & -Y_2 \\ -Y_2^2 & Y_2^2 & Y_2X_2 & -Y_2 & -X_2 \\ X_2 & X^2 & Y_2 & -N & 0 \\ -Y_2 & Y_2 & X_2 & 0 & -N \end{pmatrix} \tag{18a}$$

where, for example, $X_2^2 = \sum_{i=1}^{N} X'^{\,2}_{2i}$ and $X_2Y_2 = \sum_{i=1}^{N} X'_{2i}Y'_{2i}$.

Similarly,

$$\underline{C} = \begin{pmatrix} X_2^2 - Y_2^2 - X_1Y_2 - Y_1X_2 \\ -X_1X_2 - Y_2X_2 \\ X_2Y_2 - Y_1Y_2 \\ -X_1 - Y_2 \\ X_2 - Y_1 \end{pmatrix} \tag{18b}$$

and

$$\underline{P} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \varepsilon_x \\ \varepsilon_y \end{pmatrix} \tag{18c}$$

The solution to equation (17) may be obtained by inversion of the matrix $\underline{\underline{D}}$, or, because of the low order of the matrix, by Kramer's rule. The latter method is that used in the computer programs given in the appendices.

Let us now suppose we have obtained the solution to equation (17), i.e., we have the best fit values of α, β, γ, ε_x and ε_y as well as the value for chi-squared. Using these parameters we can calculate a value for the "true" coordinates. The equations are:

$$\underline{X}_i = \underline{\underline{A}}^{-1} X'_{1i} \tag{19a}$$

and

$$\underline{X}_i = \underline{\underline{B}}^{-1} \underline{\underline{A}}^{-1} \underline{X}'_{2i} + \underline{\varepsilon} \tag{19b}$$

A resulting "best" value for the coordinates may be obtained by a simple average; that is:

$$\overline{\underline{X}}_i = \frac{1}{2} \left(\underline{\underline{A}}^{-1} X'_{1i} + \underline{\underline{B}}^{-1} \underline{\underline{A}}^{-1} \underline{X}'_{2i} \right) \tag{20}$$

where we have already subtracted a factor $\frac{1}{2} \underline{\varepsilon}$ in order that the reference point have coordinates (0,0).† In this calculation of the best values for the coordinates the result is no longer independent of the choice of $\underline{\underline{A}}$, unless γ is zero. Here the metrologist must decide which of the three forms to use and this decision can only be based upon prior information or intuition. (The computer program given in the appendices has the option for using any of the three forms.) The set of coordinates, $\overline{\underline{X}}_i$, are still probably not in the desired system as they are in a coordinate system aligned with the machine axes. They are put into the preferred gage system, which usually has one point with a large X coordinate which is specified to have a zero Y coordinate, by a simple rotation. If care was taken in the initial alignment this rotation will be small, but this is not a

†Since the vector $\underline{\varepsilon}$ does not appear in the final solution for the coordinates its introduction may be unnecessary. This, however, would be difficult to prove as the coefficients for $\underline{\varepsilon}_x$ and $\underline{\varepsilon}_y$ do appear in the solutions for the other parameters

necessity for the algorithm to work. All that is required is that positions 1 and 2 differ by a rotation that is within about 10 sec of 90°.

3. RESULTS

This algorithm was checked in two different ways. The first check consisted of trying the program on data which was computer generated. This data is shown in Table 1 which includes the "true" values, the two sets of coordinates as seen in two positions nearly 90 degrees apart fitting the data (using option 3, i.e., splitting the metric error). The slight disagreements between the computed coordinates and parameters and the "true" values are interpreted as stemming from the truncation of the data at the microinch level. Also shown are the results of Reeve's program applied to the same data.

Some testing of this algorithm has also been done on real data obtained from the NBS 2-D ball plate measured on our Moore 5-Z coordinate measuring machine. If the scale error (γ) is set equal to zero the values obtained agree well with those obtained from using the full multiple redundancy of Reeve. These numbers are presented in Table 2.

4. CONCLUSIONS

It appears that this algorithm can be a valuable and relatively simple tool for uncovering and correcting for simple errors in machine geometry. Its advantages when compared with the complete multiple redundancy of Reeve are three-fold. First, it is simple enough to be programmed on a small computer, if the machine has the capability of double precision arithmetic.* Secondly, this algorithm includes a provision for assessing scale errors and, thirdly, the measurement method required coincides with techniques usually used by the operators of measuring machines. On the negative side, this method is definitely less flexible in terms of what kinds of measurements it requires; the desire to keep the program small enough for a minicomputer leads to necessarily stringent requirements on alignment to keep our approximations valid. Also, this method requires fewer measurements than the original algorithms which may prevent the averaging of other errors that is inherent in full multiple redundancy and the statistics used are certainly of an ad hoc nature. The fact that it gives the same answers and standard deviations as the more powerful method assures us somewhat on this latter point.

In order to make this technique more useful to a variety of measuring machine users, a program using the simple metric, equation (2), and a program with the option of choosing one or all of the other three, are provided in the appendices. They are in double precision Fortran of a vintage suitable for most compilers. The program for the metric described in the text is in Appendix D, while Appendix C contains a program for a simpler metric and Appendix B the proof that the observational equations are the same for any of the three matrices, equations (3), (4), and (5).

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*Least square fitting of this type requires taking differences of very large numbers which are often very similar in value. In coordinate measurement so many significant figures are required and differences are so small, it is doubtful that any of the programs described would work in single precision.

Table 1. Results of Programs Applied to Computer-generated Data.*

	True Values	Raw Data Position 1	Raw Data Position 2	ALBE 3 Results Option 3	Reeve Results
X_1	.000000	.000000	.000010	.000000	.000000
X_2	12.526471	12.526643	.001604	12.526471	12.526557
X_3	3.141597	3.141647	2.674759	3.141597	3.141619
X_4	.132671	.132702	11.989814	.132671	.132674
X_5	12.026450	12.026648	13.779498	12.026449	12.026532
X_6	6.936245	6.936358	7.217926	6.936245	6.936293
X_7	12.137425	12.137615	9.875151	12.137425	12.137508
X_8	1.110020	1.111044	3.762727	1.110020	1.110028
X_9	9.735164	9.735305	3.166073	9.735164	9.735231
Y_1	.000000	.000000	-.000014	.000000	.000000
Y_2	.000000	-.000091	-12.526485	-.000000	-.000000
Y_3	2.674327	2.674304	-3.141279	2.674327	2.674346
Y_4	11.989642	11.989641	-.131198	11.989642	11.989724
Y_5	13.777777	13.777683	-12.024755	13.777770	13.777864
Y_6	7.216943	7.216893	-6.935364	7.216943	7.216993
Y_7	9.873462	9.873374	-12.136215	9.873462	9.873530
Y_8	3.762542	3.762534	-1.110568	3.762542	3.762568
Y_9	3.164785	3.164715	-9.734786	3.164786	3.164807
Parameters ($X10^{-6}$)					
Alpha	4.79	--	--	4.81	4.66
Beta	-131.26	--	--	-131.24	-133.98
Gamma	13.70	--	--	13.69	--
X-offset	-14.20	--	--	-14.04	-13.67
Y-offset	9.60	--	--	9.45	10.01
Sigma	--	--	--	.5	57.5

*All dimensions are in inches and angles are in radians. Gamma is dimensionless.

Table 2. Results of Programs Applied to Real Data

	Raw Data		ALBE 3 Results	Reeve
	Position 1	Position 2	Option 3	Results
X_1	.000000	.000000	-.000000	.000000
X_2	-3.000912	3.998718	-3.000864	-3.000868
X_3	-2.000640	10.999142	-2.000515	-2.000517
X_4	-4.000689	13.998858	-4.000532	-4.000537
X_5	-.000163	14.999467	.000000	.000000
X_6	-8.001243	1.998189	-8.001196	-8.001207
X_7	-6.001024	5.998378	-6.000930	-6.000939
X_8	-9.001392	8.998041	-9.001258	-9.001270
X_9	-7.000838	12.998368	-7.000682	-7.000692
X_{10}	-15.002162	.996753	-15.002107	-15.002129
X_{11}	-13.001803	4.997135	-13.001700	-13.001718
X_{12}	-12.001441	9.997590	-12.001292	-12.001309
X_{13}	-14.002280	14.996517	-14.002106	-14.002125
Y_1	.000000	.000000	.000000	.000000
Y_2	3.999098	3.001472	3.999145	3.999151
Y_3	10.999378	2.002176	10.999406	10.999422
Y_4	13.999388	4.002653	13.999428	13.999447
Y_5	14.999390	.002266	14.999408	14.999429
Y_6	1.999283	8.001495	1.999381	1.999384
Y_7	5.999186	6.001825	5.999259	5.999268
Y_8	8.999253	9.002606	8.999363	8.999376
Y_9	12.999295	7.002651	12.999380	12.999398
Y_{10}	.998845	15.002257	.999012	.999014
Y_{11}	4.998944	13.002443	4.999086	4.999092
Y_{12}	9.999228	12.002806	9.999367	9.999381
Y_{13}	14.998424	14.004399	14.998583	14.998604
Parameters ($X10^{-6}$)				
Alpha	--	--	-.67	-.66
Beta	--	--	-141.27	-141.50
Gamma	--	--	2.82	--
X-offset	--	--	-14.30	0
Y-offset	--	--	-17.58	0
Sigma	--	--	16.9	17.0

APPENDIX A: A THREE PARAMETER FORM

A simple form for the machine metric is that described in the text, that is

$$\underline{\underline{A}} = \begin{pmatrix} 1+\gamma & -\alpha \\ 0 & 1 \end{pmatrix} \tag{A1}$$

This metric can be used and a simpler computation (with a shorter program) done by neglecting the offsets ε_x and ε_y. The observational equations are then

$$X'_{1i} = -(\beta + \alpha)\ X'_{2i} - (1+\gamma)\ Y'_{2i} \tag{A2a}$$

and

$$Y'_{1i} = (1-\gamma) + (\alpha - \beta)\ Y'_{2i} \tag{A2b}$$

The linear equations at the minimum in chi-squared are:

$$\begin{pmatrix} 0 & 2X_2Y_2 & Y_2^2+X_2^2 \\ X_2^2 & X_2^2 & Y_2X_2 \\ -Y_2^2 & Y_2^2 & Y_2X_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \doteq \begin{pmatrix} X_2^2-Y_2^2-X_1Y_2-Y_1X_2 \\ -X_1X_2 \\ X_2Y_2-Y_1Y_2 \end{pmatrix} \tag{A3}$$

A Fortran program for the solution of A3 appears as Appendix C. Table A1 shows the results of the program on the dummy data described in the text.

Table A1 also shows the results on the real ball plate data previously described, and comparison of these results with those obtained using Reeve's full multiple redundancy. The large standard deviation in the Reeve result on the dummy data is due to the inclusion of a length scale error, γ, of 13.7 ppm when the data were generated.

Table A1. Results of Programs Applied to Computer-generated Data, with Offsets E_x and E_y Neglected.*

	ALBE 2 Results on Dummy Data	Reeve Results	ALBE 2 Results on Real Data	Reeve Results
X_1	.000007	.000000	.000000	.000000
X_2	12.526483	12.526557	-3.000862	-3.000868
X_3	3.141604	3.141619	-2.000514	-2.000517
X_4	.132673	.132672	-4.000530	-4.000537
X_5	12.026455	12.026532	-.000000	.000000
X_6	6.936251	6.936293	-8.001193	-8.001207
X_7	12.137433	12.137508	-6.000928	-6.000939
X_8	1.111026	1.111028	-9.001253	-9.001270
X_9	9.735171	9.735231	-7.000678	-7.000692
X_{10}			-15.002101	-15.002129
X_{11}			-13.001694	-13.001718
X_{12}			-12.001287	-12.001309
X_{13}			-14.002099	-14.002125
Y_1	-.000005	.000000	.000000	.000000
Y_2	-.000000	-.000000	3.999144	3.999151
Y_3	2.674324	2.674346	10.999401	10.999422
Y_4	11.989642	11.989724	13.999422	13.999447
Y_5	13.777775	13.777864	14.999401	14.999429
Y_6	7.216943	7.216993	1.999381	1.999384
Y_7	9.873466	9.873530	5.999257	5.999268
Y_8	3.762539	3.762568	8.999460	8.999376
Y_9	3.164785	3.164807	12.999374	12.999398
Y_{10}			.999012	.999014
Y_{11}			4.999084	4.999092
Y_{12}			9.999362	9.999381
Y_{13}			14.998577	14.998604
Parameters (10^{-6} inches or radiams)				
Alpha	5.03	4.66	-.81	-.66
Beta	-130.55	-133.98	-140.40	-141.50
Gamma	12.99	--	3.69	--
Sigma	7.0	57.5	17.6	17.0

*All dimensions are in inches and angles are in radians.

Appendix B

EQUIVALENCE OF THE OBSERVATIONAL EQUATIONS FOR THE THREE FORMS OF METRIC ERROR.

Three logical choices for a machine metric with scale errors were given in the text. They were

$$\underline{\underline{A}}_x = \begin{pmatrix} 1+\gamma & -\alpha \\ 0 & 1 \end{pmatrix} \qquad \text{(B1a)}$$

$$\underline{\underline{A}}_y = \begin{pmatrix} 1 & -\alpha \\ 0 & 1-\gamma \end{pmatrix} \qquad \text{(B1b)}$$

and

$$\underline{\underline{A}} = \begin{pmatrix} 1+\frac{\gamma}{2} & -\alpha \\ 0 & 1-\frac{\gamma}{2} \end{pmatrix} \qquad \text{(B1c)}$$

The basic observational equation is, in matrix notation,

$$\underline{X}'_{1i} = \underline{\underline{A}}\,\underline{\underline{B}}^{-1}\,\underline{\underline{A}}^{-1}\,\underline{X}'_{2i} + \underline{\varepsilon} = \underline{\underline{C}}\,\underline{X}'_{2i} + \underline{\varepsilon} \qquad \text{(B2)}$$

To show that the three metrics above yield the same observational equations we need only to show that

$$\underline{\underline{C}} = \underline{\underline{A}}_x\,\underline{\underline{B}}^{-1}\,\underline{\underline{A}}_x^{-1} = \underline{\underline{A}}_y\,\underline{\underline{B}}^{-1}\,\underline{\underline{A}}_y = \underline{\underline{A}}\,\underline{\underline{B}}^{-1}\,\underline{\underline{A}}^{-1} \qquad \text{(B3)}$$

For the case where the metric is given by $\underline{\underline{A}}_x$ we have already shown in the text that

$$\underline{\underline{C}} = \begin{pmatrix} -(\beta+\alpha) & -(1+\gamma) \\ (1-\gamma) & \alpha-\beta \end{pmatrix} \qquad \text{(B4)}$$

when

$$\underline{\underline{B}}^{-1} = \begin{pmatrix} -\beta & -1 \\ 1 & -\beta \end{pmatrix}$$

The inverse forms for $\underline{\underline{A}}_y$ and $\underline{\underline{A}}$ are

$$\underline{\underline{A}}_y^{-1} = \begin{pmatrix} 1 & \alpha \\ 0 & 1+\gamma \end{pmatrix} \tag{B5a}$$

and

$$\underline{\underline{A}}^{-1} = \begin{pmatrix} 1-\gamma/2 & \alpha \\ 0 & 1+\gamma/2 \end{pmatrix} \tag{B5b}$$

Substitution of either (B5a) or (B5b) into (B3) will yield, to first order in the parameters, $\underline{\underline{C}}$, as given in equation (B4).

```
      SUBROUTINE ALBE2(X,Y,G,SIGMA,NPTS)
C
C X AND Y ARE THE ARRAYS FOR THE DATA.  THE FIRST SUBSCRIPT IN EACH
C IS USED TO DETERMINE THE POSITION OF THE GAGE (1 OR 2). POS 2
C IS ABOUT 90 DEGREES CLOCKWISE, VIEWED FROM THE TOP, FROM
C POSITION 1.
C    G(1)=NONORTHOGONALITY  ANGLE, IN RADIANS
C    G(2)=ROTATION DIFFERENCE FROM 90 DEGREES, RADIANS
C    G(3)=METRIC ERROR (ASSUMED EQUAL BETWEEN X AND Y)
C    SIGMA=RMS STANDARD DEVIATION IN COORDINATES.  UNITS ARE THE SAME
C    AS THOSE USED IN X AND Y, MULTIPLIED BY 1E06.
C    NPTS=NUMBER OF GAGE POINTS MEASURED. DIMENSIONED FOR 50.
C
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION X(3,50),Y(3,50),A(3,3),D(3,3),C(3),G(3)
C
C    SET SUMS TO ZERO
C
      Y22=0.D0
      X22=0.D0
      XY12=0.D0
      XY21=0.D0
      XY22=0.D0
      XX12=0.D0
      YY12=0.D0
      DO 100 I=1,NPTS
      Y22=Y22+Y(2,I)**2
      X22=X22+X(2,I)**2
      XY12=XY12+X(1,I)*Y(2,I)
      XY21=XY21+X(2,I)*Y(1,I)
      XY22=XY22+X(2,I)*Y(2,I)
      XX12=XX12+X(1,I)*X(2,I)
100   YY12=YY12+Y(1,I)*Y(2,I)
C
C    SET UP MATRIX
C
      A(1,1)=0.D0
      A(1,2)=2.D0*XY22
      A(1,3)=Y22+X22
      A(2,1)=X22
      A(2,2)=X22
      A(2,3)=XY22
      A(3,1)=-Y22
      A(3,2)=Y22
      A(3,3)=XY22
      C(1)=-(XY12+XY21+Y22-X22)
      C(2)=-XX12-XY22
      C(3)=-YY12+XY22
C
C    D3 CALCULATES DETERMINANT OF A
C
      DD=D3(A)
C    IF MATRIX IS SINGULAR, PRINT MESSAGE
      IF(DD.EQ.0.D0) WRITE(6,20)
      DO 150 I=1,3
      DO 120 J=1,3
      DO 120 K=1,3
```

```
120    D(J,K)=A(J,K)
       DO 130 L=1,3
130    D(L,I)=C(L)
150    G(I)=D3(D)/DD
C
C  CALCULATE G(1) THROUGH G(3) FOR RETURN TO MAIN PROGRAM
C
C
C   COMPUTE CHISQUARE
C
       CHISQ=0.D0
       DO 200 I=1,NPTS
       XC=-(G(1)+G(2))*X(2,I)-(1.D0+G(3))*Y(2,I)
       YC=(1.D0-G(3))*X(2,I)+(G(1)-G(2))*Y(2,I)
       CHISQ=CHISQ+(X(1,I)-XC)**2+(Y(1,I)-YC)**2
       X(3,I)=(X(1,I)+XC)*(1.D0-G(3))/2.D0+(Y(1,I)+YC)*G(1)/2.D0
       Y(3,I)=(Y(1,I)+YC)/2.D0
200    CONTINUE
       FREE=2.D0*NPTS-3.D0
       SIGMA=1.D+06*DSQRT(CHISQ/FREE)
20     FORMAT(1X,' MATRIX OF COE. IS SINGULAR')
       RETURN
       END
       FUNCTION D3(A)
       IMPLICIT REAL*8 (A-H,O-Z)
       DIMENSION A(3,3)
       D3=A(1,1)*A(2,2)*A(3,3)+A(1,2)*A(2,3)*A(3,1)+A(1,3)*A(2,1)*A(3,2)
      C-A(3,1)*A(2,2)*A(1,3)-A(3,2)*A(2,3)*A(1,1)-A(3,3)*A(2,1)*A(1,2)
       RETURN
       END
```

```
      SUBROUTINE ALBE3(X,Y,G,SIGMA,NPTS,NOPT)
C
C X AND Y ARE THE ARRAYS FOR THE DATA.  THE FIRST SUBSCRIPT IN EACH
C IS USED TO DETERMINE THE POSITION OF THE GAGE (1 OR 2). POS 2
C IS ABOUT 90 DEGREES CLOCKWISE, VIEWED FROM THE TOP, FROM
C POSITION 1
C    G(1)=NONORTHOGONALITY ANGLE, RADIANS
C    G(2)=ROTATION DIFFERENCE FROM 90 DEGREES, RADIANS
C    G(3)=METRIC ERROR
C    G(4)=X-OFFSET
C    G(5)=Y-OFFSET
C    SIGMA=RMS STANDARD DEVIATION IN COORDINATES.  UNITS ARE THE SAME
C     AS THOSE USED IN X AND Y
C    NPTS=NUMBER OF GAGE POINTS MEASURED. DIMENSIONED FOR 50.
C    NOPT=OPTION TO CHOOSE FORM OF SCALE ERROR:
C      1=ALL ERROR IS IN X AXIS
C      2=ALL ERROR IS IN Y AXIS
C      3=ERROR IS SPLIT BETWEEN X AND Y AXIS
C      4=THERE IS NO METRIC ERROR
C
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION X(2,50),Y(2,50),A(5,5),D(5,5),C(5),G(5)
C
C    SET SUMS TO ZERO
C
      PN=NPTS
      IF(NOPT-2) 60,61,62
60    OPT=1.D0
      GO TO 63
61    OPT=0.D0
      GO TO 63
62    OPT=0.5D0
63    CONTINUE
      X1=0.D0
      Y1=0.D0
      Y2=0.D0
      X2=0.D0
      Y22=0.D0
      X22=0.D0
      XY12=0.D0
      XY21=0.D0
      XY22=0.D0
      XX12=0.D0
      YY12=0.D0
C
C    DO SUMS NEEDED
C
      DO 100 I=1,NPTS
      X2=X2+X(2,I)
      Y2=Y2+Y(2,I)
      X1=X1+X(1,I)
      Y1=Y1+Y(1,I)
      Y22=Y22+Y(2,I)**2
      X22=X22+X(2,I)**2
      XY12=XY12+X(1,I)*Y(2,I)
      XY21=XY21+X(2,I)*Y(1,I)
      XY22=XY22+X(2,I)*Y(2,I)
```

```
      XX12=XX12+X(1,I)*X(2,I)
100   YY12=YY12+Y(1,I)*Y(2,I)
C
C
C    SET UP MATRIX FOR SOLUTION
C
      A(1,1)=0.D0
      A(1,2)=2.D0*XY22
      A(1,3)=Y22+X22
      A(1,4)=-Y2
      A(1,5)=-X2
      A(2,1)=X22
      A(2,2)=X22
      A(2,3)=XY22
      A(2,4)=-X2
      A(2,5)=0.D0
      A(3,1)=-Y22
      A(3,2)=Y22
      A(3,3)=XY22
      A(3,4)=0.D0
      A(3,5)=-Y2
      A(4,1)=X2
      A(4,2)=X2
      A(4,3)=Y2
      A(4,4)=-RN
      A(4,5)=0.D0
      A(5,1)=-Y2
      A(5,2)=Y2
      A(5,3)=X2
      A(5,4)=0.D0
      A(5,5)=-RN
      C(1)=-(XY12+XY21+Y22-X22)
      C(2)=-XX12-XY22
      C(3)=-YY12+XY22
      C(4)=-X1-Y2
      C(5)=X2-Y1
      DD=DETERM(A,5)
C    WRITE ERROR MESSAGE IF MATRIX IS SINGULAR
      IF(DD.EQ.0.D0) WRITE(6,20)
      DO 150 I=1,5
      DO 120 J=1,5
      DO 120 K=1,5
120   D(J,K)=A(J,K)
      DO 130 L=1,5
130   D(L,I)=C(L)
C    CALCULATE G(1) THROUGH G(5) FOR RETURN TO MAIN PROGRAM
150   G(I)=DETERM(D,5)/DD
C
C    CALCULATE CHISQ
C
      CHISQ=0.D0
      DO 200 I=1,NPTS
      XC=-(G(1)+G(2))*X(2,I)-(1.D0+G(3))*Y(2,I)+G(4)
      YC=(1.D0-G(3))*X(2,I)+(G(1)-G(2))*Y(2,I)+G(5)
      CHISQ=CHISQ+(X(1,I)-XC)**2+(Y(1,I)-YC)**2
      XC=((X(1,I)+XC)/2.D0)-G(4)/2
      YC=((Y(1,I)+YC)/2.D0)-G(5)/2
      IF (NOPT.EQ.4) G(3)=0.D0
```

```
      X(3,I)=XC*(1.D0-OPT*G(3))+YC*G(1)
      Y(3,I)=(1.D0+(1.D0-OPT)*G(3))*YC
200   CONTINUE
      FREE=2.D0*NPTS-5.D0
      SIGMA=1.D+06*DSQRT(CHISQ/FREE)
20    FORMAT(1X,' MATRIX OF COE. IS SINGULAR')
      RETURN
      END
      FUNCTION DETERM(AA,NORDER)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION ARRAY(5,5),AA(5,5)
10     DETERM=1.D0
      DO 45 J=1,NORDER
      DO 45 K=1,NORDER
45    ARRAY(J,K)=AA(J,K)
11    DO 50 K=1,NORDER
      IF(ARRAY(K,K)) 41,21,41
21    DO 23 J=K,NORDER
      IF(ARRAY(K,J)) 31,23,31
23    CONTINUE
      DETERM=0.D0
      GO TO 60
31    DO 34 I=K,NORDER
      SAVE=ARRAY(I,J)
      ARRAY(I,J)=ARRAY(I,K)
34    ARRAY(I,K)=SAVE
      DETERM=-DETERM
41    DETERM=DETERM*ARRAY(K,K)
      IF(K-NORDER) 43,50,50
43    K1=K+1
      DO 46 I=K1,NORDER
      DO 46 J=K1,NORDER
46    ARRAY(I,J)=ARRAY(I,J)-ARRAY(I,K)*ARRAY(K,J)/ARRAY(K,K)
50    CONTINUE
60    RETURN
      END
```